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An Analytical Solution for Creeping Flow in a Narrow Gap between Two Surfaces Created by Injected Flux

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Keynotes: creeping flow, manifolds, laplace beltrami, sources

1. Abstract

A viscous flow through a narrow gap between surfaces is free of the inertial effects and can be characterized as a creeping flow. Since every physical set-up takes place in the R^3 space, while the mathematical solutions are on two dimensional surfaces, it is required to do some adjustments for obtaining the locally equivalent problem without dependence on the perpendicular direction. Averaging the velocity along the gap e enables us to consider as a 2D problem which is governed by the Poisson equation. This study is related to a flow between surfaces, such as two coaxial cylinders (FCC), two co-axial cones (FCN) and concentric spheres (FCS), namely a shell. The driving force is supplied by several point sources, which creates a pressure field that is investigated and an analytical solution is obtained. For the three geometries, particular cases of truncation ends were studied which boundary conditions flux considerations and have been taken into account, where an injected flux into a finite volume must summed up to zero.

Keywords: Creeping-Flow**,** manifolds, Injected flux, sources

2. Introduction

This paper presents solutions for some physical problems with different geometries with an injected flux inside the gap between two surfaces for each case. Such flow is encountered in the process industry, in various of lubrication or heat transfer systems, where through the geometric device, there are injected fluxes. The cold fluid is streamed inside the core. Furthermore, it can be encountered in systems that are based on diffusion of gas through membrane inside a liquid that flows in a gap between two membranes, as in the physiologic area, where there is a blood flow around the alveoli. The solution for the flow between two Concentric Spheres (FCS) is presented by a new available solution that was recently proposed by Eyal and Goldstein, 2018. They found the fundamental solution for the Laplace operator in S^d by using the solution in Euclidean space as an intermediate stage for obtaining a solution in the desired surface. In this study we wish to utilize this solution by substituting $D = 3$ and the solution considers the flow on a sphere at S^2, as a particular case, in order to solve the Poisson Equation. Bogomolov, 1977 obtained the solution for the FCS problem, either, based on the Weiss' theorem at R^3, where the flow is created by distributed vortices, sinks and sources, but without reference to the pressure distribution. He introduced the flow equation in a narrow gap with similar results as in the Helle Shaw problem. A solution of a non linear of the N-S equation was obtained for flow inside rotating concentric spheres, with a single source. Analytical solutions for the flow between two co-axial cylinders (FCC flow) and two co-axial cones (FCN flow), were derived in this paper after considering the geometric shape and the characteristic periodic properties of their boundary conditions. Noting that, for an infinite cylinder, the solution reveals an interesting phenomena, such as Saddle-Points that exist for some critical values of the potential. The physical significance for this is for cases where detachment of the Equi-Potential contours occurs (bifurcation of level line).

FCC Flow: Equi-Potential Contour Lines in two Co-axial Cylinders

This section demonstrates one of the six different configurations that are studied in this research (e.g., an infinite cylinder, open-open truncated, etc.,).

Fig.1 demonstrates the Equi-Pressure and streamlines of the field flow for a semi-infinite one sided open cylinder affected by a source that is located at $\phi_{source} = 0$; Z_{source} and its image at $\phi_{image} = 0$; $Z_{image} = -Z_{source}$. Fig.1a shows that the Equi-Pressure line, as predicted, at the truncated open end encircles the cylinder which implies that the velocity is axial, i.e. directed parallel to Z and flows out from the cylinder. As long as the distance Z increases the density of the Equi-Pressure line decreases implying for a decreased velocity as well as the pressure. As can be seen, the direction of the Equi-Pressure line is parallel to Z at the point B and the direction of the velocity is $-\widehat{\emptyset}$, which is the angle that separate for the forward/backward ow direction. An explanation for the flow direction is presented in Fig.1b, where stream lines are shown at a sliced cylinder. As explained above, for a net zero of sources (the source is superimposed with its image), above the sources, the net flux is zero, result in a forward ow as well as backward ow. Fig.1c demonstrates the direction of the flow field $\hat{v}$ far away from the source as function of ϕ. At point A for $\phi = 0$, the fluid is characterized with a forward flow and propagates along Z. For $0 < \widehat{\emptyset} < \frac{\pi}{2}$, the velocity changes gradually its direction from The $\hat{Z}$ direction to $\widehat{\emptyset}$. Increasing more ϕ up to _ (point C, the lower surface of the cylinder), a backward flow is observed for keeping the net zero flux over $Z = const$. The backward ow is disappeared and at point B the flow is directed to $-\widehat{\emptyset}$ and increasing ϕ the fluid climbs up to the top surface and v_z increases either.

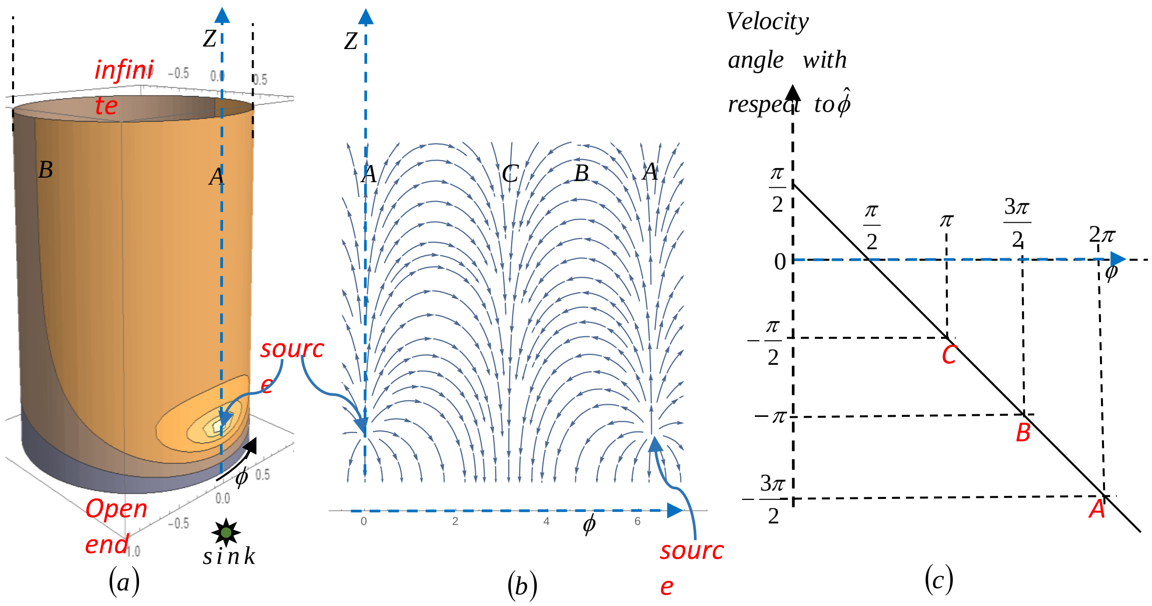

Figure 1: *Asymptotic behaviour of the ow on a semi-infinite open cylinder affected by one source: (a) Equi-pressure contours and (b) Streamlines of the field ow; (c)The angle of the velocity vector with respect to* $\widehat{\emptyset}$

$$p = p_s + p_c = \underbrace{-\frac{q}{4\pi A}\ln\left(\frac{\cosh^2\left(\frac{Z-Z_{source}}{2}\right)-\cos^2\left(\frac{\phi-\phi_{source}}{2}\right)}{\cosh^2\left(\frac{Z-Z_{image}}{2}\right)-\cos^2\left(\frac{\phi-\phi_{image}}{2}\right)}\right)}_{p_s} + p_c$$

The above equation presents the pressure distribution on such flow, where there are the contribution of the source and its image for obeying the boundary condition at the open end.

FCS Flow: Flow Field on Sphere

The solutions for the six cases that are shown in Fig.2 and demonstrates the all possibilities for the flows distribution as following:

Cases a,c and d (full sphere, truncated closed cap and doubly closed truncated accordingly) are characterized as a finite surface, which binding a net zero flux (achieving with sources and sinks with the same magnitude). Cases *b* is shown in Fig.2b, *e* and *f* are open truncated and the fluid flows out from the surface and consequently to that, there is no need for a net zero sources in this domain. Fig.2*b; e; f* demonstrates the Equi-Pressure that at the open end, an Equi-Pressure line surrounds it, which physically means that the fluid flows out to a constant prescribed pressure. Actually, it can be solved easily analytically by locating images at the poles.

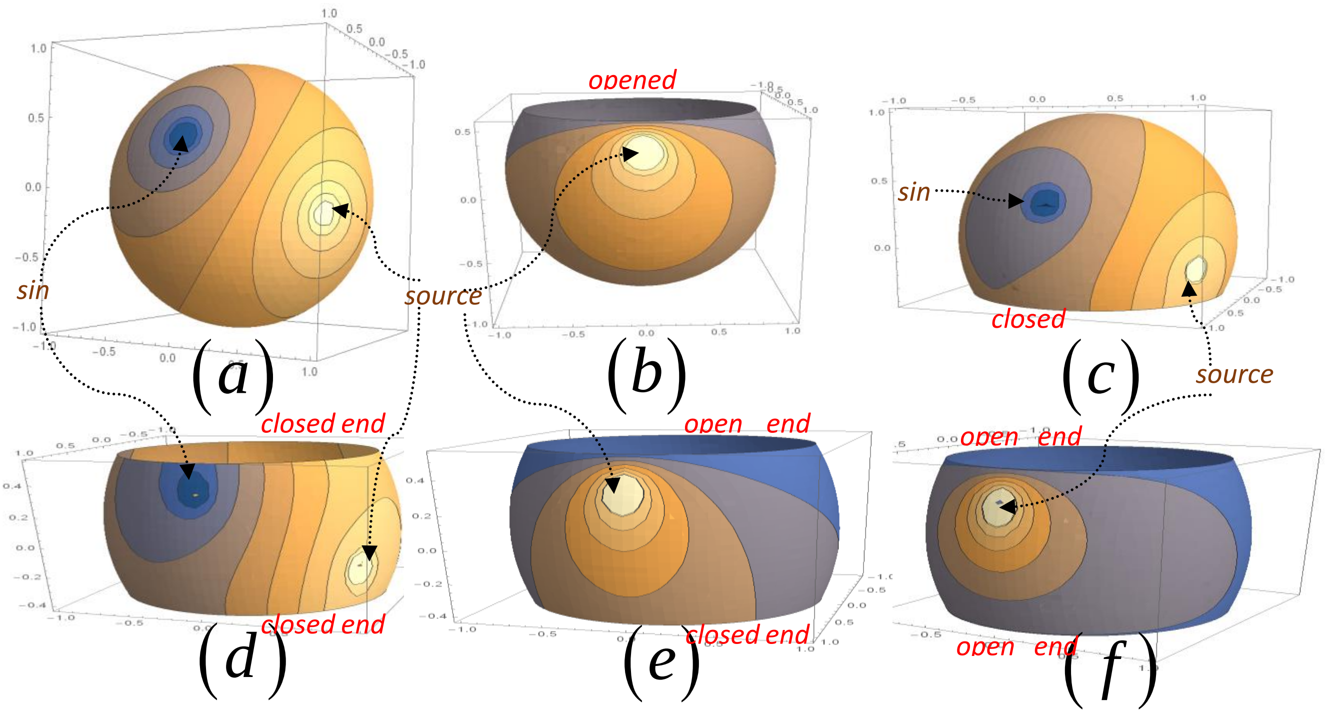

Figure 2: *Pressure distribution of FCS ow and its special cases for embedded source/sink: (a) A source and a sink placed on a sphere (b) An open cap (one sided truncated sphere): the liquid emanated from the source flows out freely through the boundary.(c) A closed cap: a source and a sink are embedded inside ; (d) Two sided truncated closed sphere affected by a source and a sink. (e) Two sided truncated sphere, one is open and the other is closed affected by one source. (f) Two sided truncated sphere, both sides are open affected one source.*

$$\begin{aligned} p(\vec{R}) &= \tfrac{1}{4\pi A}\int dQ \tfrac{1}{\|\vec{R}-\hat{R}'R'\|} = \tfrac{1}{4\pi A}\tfrac{q}{a}\int_0^\infty dR' \tfrac{1}{\|\vec{R}-\hat{R}'R'\|} = -\tfrac{1}{A4\pi a}\sum_{i=0}^{N-1} q_i \ln(1-\hat{R}\cdot\hat{R}_i) = \\ &= -\tfrac{1}{2\pi a A}\sum_{i=0}^{N-1} q_i \ln(\sin(\theta_{\hat{R},\hat{R}_i}/2)) = -\tfrac{1}{2\pi a A}\sum_{i=0}^{N-1} q_i \ln(\ell_i/R) \end{aligned}$$

Noting that, similarly for the both geometries FCC and FCS, equations were explored for the flow between co-axial cones either, but will not be presented in this abstract.

3. References

[1] V. A. Bogomolov, \Dynamics of vorticity on a sphere," *Fluid Dyn*, vol. 6, p. 863, 1977.
[2] C. L.M.Milne-Thompson, *Theoretical Hydrodynamics*. Macmilan Education LTD, 1986.
[3] G. Batchelor, *An Introduction to FLUID DYNAMICS*. Cambridge university press, 2002.
[4] O. Eyal and A. Goldstein, \The fundamental solution for the laplace-beltrami operator on s^n using dimensional analysis," *To appear*, 2018.
[5] G. B. Arfken and H. J. Weber, *Mathematical methods for Physicists*. Harcourt Academic Press, 3 ed., 1996.
[6] P. Henrici, *Applied and Computational Complex Analysis*, vol. 1. Wiley & Sons, 1974.
[7] O. Eyal and E. Raz, \Using dimensional analysis to _nd _elds created by charges or currents distributed on self-scaled geometric shapes," *Eur. J. Phys.*, vol. 37, p. 45205, 2016.
[8] H. S. Cohl, \Fundamental solution of laplace's equation in hyperspherical geometry," *SIGMA*, vol. 7, p. 108, 2011.
[9] J.-M. Caillol, \Electrostatics on the sphere with applications to monte carlo simulations of two dimensional polar uids,"
vol. 142, 01 2015.
[10] D. G. Crowdy, \Stuart vortices on a sphere," *J. Fluid Mech*, vol. 498, pp. 381{402, 2004.